# INSERTION AND DELETION TOLERANCE OF POINT PROCESSES

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Abstract. We develop a theory of insertion and deletion tolerance for point processes. A process is insertion-tolerant if adding a suitably chosen random point results in a point process that is absolutely continuous in law with respect to the original process. This condition and the related notion of deletion-tolerance are extensions of the so-called finite energy condition for discrete random processes. We prove several equivalent formulations of each condition, including versions involving Palm processes. Certain other seemingly natural variants of the conditions turn out not to be equivalent. We illustrate the concepts in the context of a number of examples, including Gaussian zero processes and randomly perturbed lattices, and we provide applications to continuum percolation and stable matching.

## 1. Introduction

Let $\Pi$ be a point process on $\mathbb{R}^d$. Point processes will always be assumed to be simple and locally finite. Let $\prec$ denote absolute continuity in law; that is, for random variables $X$ and $Y$ taking values in the same measurable space, $X \prec Y$ if and only if $\mathbb{P}(Y \in \mathcal{A}) = 0$ implies $\mathbb{P}(X \in \mathcal{A}) = 0$ for all measurable $\mathcal{A}$. Let $\mathfrak{B}$ denote the Borel $\sigma$-algebra on $\mathbb{R}^d$ and let $\mathcal{L}$ be Lebesgue measure. We say that $\Pi$ is **insertion-tolerant** if for every $S \in \mathfrak{B}$ with $\mathcal{L}(S) \in (0,\infty)$, if $U$ is uniformly distributed on $S$ and independent of $\Pi$, then

$$\Pi + \delta_U \prec \Pi,$$

where $\delta_x$ denotes the point measure at $x \in \mathbb{R}^d$.

Let $\mathbb{M}$ denote the space of simple point measures on $\mathbb{R}^d$. The support of a measure $\mu \in \mathbb{M}$ is denoted by

$$[\mu] := \left\{ y \in \mathbb{R}^d : \mu(\{y\}) = 1 \right\}.$$

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A **$\Pi$-point** is an $\mathbb{R}^d$-valued random variable $Z$ such that $Z \in [\Pi]$ a.s. A **finite subprocess** of $\Pi$ is a point process $\mathcal{F}$ such that $\mathcal{F}(\mathbb{R}^d) < \infty$ and $[\mathcal{F}] \subseteq [\Pi]$ a.s. We say that $\Pi$ is **deletion-tolerant** if for any $\Pi$-point $Z$ we have

$$\Pi - \delta_Z \prec \Pi.$$

For $S \in \mathfrak{B}$ we define the restriction $\mu|_S$ of $\mu \in \mathbb{M}$ to $S$ by

$$\mu|_S(A) := \mu(A \cap S), \qquad A \in \mathfrak{B}.$$

We will prove the following equivalences for insertion-tolerance and deletion-tolerance.

**Theorem 1** (Deletion-tolerance)**.** *Let $\Pi$ be a point process on $\mathbb{R}^d$. The following are equivalent.*

*(i) The point process $\Pi$ is deletion-tolerant.*
*(ii) For any finite subprocess $\mathcal{F}$ of $\Pi$, we have $\Pi - \mathcal{F} \prec \Pi$.*
*(iii) For all $S \in \mathfrak{B}$ with finite Lebesgue measure, $\Pi|_{S^c} \prec \Pi$.*

**Theorem 2** (Insertion-tolerance)**.** *Let $\Pi$ be a point process on $\mathbb{R}^d$. The following are equivalent.*

*(i) The point process $\Pi$ is insertion-tolerant.*
*(ii) For any Borel sets $S_1, \dots, S_n$ of positive finite Lebesgue measure, if $U_i$ is a uniformly random point in $S_i$, with $U_1, \dots, U_n$ and $\Pi$ all independent, then $\Pi + \sum_{i=1}^n \delta_{U_i} \prec \Pi$.*
*(iii) If $(X_1, \dots, X_n)$ is a random vector in $(\mathbb{R}^d)^n$ that admits a conditional law given $\Pi$ that is absolutely continuous with respect to Lebesgue measure a.s., then $\Pi + \sum_{i=1}^n \delta_{X_i} \prec \Pi$.*

In fact we will prove a stronger variant of Theorem 2, in which (ii),(iii) are replaced with a condition involving the insertion of a *random* finite number of points.

We say that a point process is **translation-invariant** if it is invariant in law under all translations of $\mathbb{R}^d$. In this case further equivalences are available as follows.

**Proposition 3.** *A translation-invariant point process $\Pi$ on $\mathbb{R}^d$ is insertion-tolerant if and only if there exists $S \in \mathfrak{B}$ with $\mathcal{L}(S) \in (0, \infty)$ such that, if $U$ is uniformly distributed in $S$ and independent of $\Pi$, then $\Pi + \delta_U \prec \Pi$.*

Let $\Pi$ be a translation-invariant point process with finite intensity; that is, $\mathbb{E}\Pi([0,1]^d) < \infty$. We let $\Pi^*$ be its *Palm version.* See Section 4 or [13, Chapter 11] for a definition. Informally, one can regard $\Pi^*$ as the point process $\Pi$ conditioned to have a point at the origin.

**Theorem 4.** *Let $\Pi$ be a translation-invariant ergodic point process of finite intensity on $\mathbb{R}^d$ and let $\Pi^*$ be its Palm version. The following are equivalent.*

*(i) The point process $\Pi$ is insertion-tolerant.*
*(ii)* $\Pi + \delta_0 \prec \Pi^*$.

Condition (1) below appears to be the natural analogue of Theorem 4 (ii) for deletion-tolerance. However, it is only sufficient and not necessary for deletion-tolerance.

**Theorem 5.** *Let $\Pi$ be a translation-invariant point process of finite intensity on $\mathbb{R}^d$ and let $\Pi^*$ be its Palm version. If*

$$\Pi^* - \delta_0 \prec \Pi, \tag{1}$$

*then $\Pi$ is deletion-tolerant.*

In Section 2, Example 3 shows that a deletion-tolerant process need not satisfy (1), while Example 6 shows that the natural analogue of Proposition 3 fails for deletion-tolerance.

**Remark 1** (More general spaces)**.** *Invariant point processes and their Palm versions can be defined on more general spaces than $\mathbb{R}^d$. See* [3, 12, 14, 16, 17] *for more information. For concreteness and simplicity, we have chosen to state and prove Theorems 1, 2, 4 and 5 in the setting of $\mathbb{R}^d$, but they can easily be adapted to any complete separable metric space endowed with: a group of symmetries that acts transitively and continuously on it, and the associated Haar measure. We will make use of this generality when we discuss Gaussian zero processes on the hyperbolic plane in Proposition 13.* $\Diamond$

Next we will illustrate some applications of insertion-tolerance and deletion-tolerance in the contexts of continuum percolation and stable matchings. We will prove generalizations of earlier results.

The Boolean continuum percolation model for point processes is defined as follows (see [19]). Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$. For $R > 0$ and $\mu \in \mathbb{M}$, consider the set $\mathcal{O}(\mu) := \cup_{x\in[\mu]} B(x, R)$, where $B(x, R) := \left\{y \in \mathbb{R}^d : \|x - y\| < R\right\}$ is the open ball of radius $R$ with center $x$. We call $\mathcal{O}(\mu)$ the **occupied region**. The connected components of $\mathcal{O}(\mu)$ are called **clusters**.

**Theorem 6** (Continuum percolation)**.** *Let $\Pi$ be a translation-invariant ergodic insertion-tolerant point process on $\mathbb{R}^d$. For any $R > 0$, the occupied region $\mathcal{O}(\Pi)$ has at most one unbounded cluster a.s.*

The proof of Theorem 6 is similar to the uniqueness proofs in [19, Chapter 7] which in turn are based on the argument of Burton and Keane [1].

Next we turn our attention to stable matchings of point processes (see [9] for background). Let $\mathcal{R}$ and $\mathcal{B}$ be ('red' and 'blue') point processes on $\mathbb{R}^d$ with finite intensities. A **one-colour matching scheme** for $\mathcal{R}$ is a point process $\mathcal{M}$ of unordered pairs $\{x,y\} \subset \mathbb{R}^d$ such that almost surely $[\mathcal{M}]$ is the edge set of a simple graph $([\mathcal{R}],[\mathcal{M}])$ in which every vertex has degree exactly one. Similarly, a **two-colour matching scheme** for $\mathcal{R}$ and $\mathcal{B}$ is a point process $\mathcal{M}$ of unordered pairs $\{x,y\} \subset \mathbb{R}^d$ such that almost surely, $[\mathcal{M}]$ is the edge set of a simple bipartite graph $([\mathcal{R}],[\mathcal{B}],[\mathcal{M}])$ in which every vertex has degree exactly one. In either case we write $\mathcal{M}(x) = y$ if and only if $\{x,y\} \in [\mathcal{M}]$. In the one-colour case, we say that a matching scheme is **stable** if almost surely there do not exist distinct points $x, y \in [\mathcal{R}]$ satisfying

$$\|x-y\| < \min\{\|x-\mathcal{M}(x)\|, \|y-\mathcal{M}(y)\|\}, \tag{2}$$

while in the two-colour case we say that a matching scheme is **stable** if almost surely there do not exist $x \in [\mathcal{R}]$ and $y \in [\mathcal{B}]$ satisfying (2). These definitions arise from the concept of stable marriage as introduced by Gale and Shapley [5].

It is proved in [9] that stable matching schemes exist and are unique for point processes that satisfy certain mild restrictions, as we explain next. Let $\mu \in \mathbb{M}$. We say that $\mu$ has a **descending chain** if there exist $x_1, x_2, \ldots \in [\mu]$ with

$$\|x_{i-1} - x_i\| > \|x_i - x_{i+1}\| \text{ for all } i.$$

We say that $\mu$ is **non-equidistant** if for all $x, y, u, v \in [\mu]$ such that $\{x,y\} \neq \{u,v\}$ and $x \neq y$ we have $\|x-y\| \neq \|u-v\|$. The following fact are proved in [9, Proposition 9]. Suppose that $\mathcal{R}$ is a translation-invariant point process on $\mathbb{R}^d$ with finite intensity that almost surely is non-equidistant and has no descending chains. Then there exists a one-colour stable matching scheme which is an isometry-equivariant factor of $\mathcal{R}$; this matching scheme may be constructed by a simple procedure of iteratively matching, and removing, mutually-closest pairs of $\mathcal{R}$-points; furthermore, any two one-colour stable schemes agree almost surely [9, Proposition 9]. In this case we refer to the above-mentioned scheme as *the* one-colour stable matching scheme. Similarly, in the two-colour case, let $\mathcal{R}$ and $\mathcal{B}$ be point processes on $\mathbb{R}^d$ of equal finite intensity, jointly invariant and ergodic under translations, and suppose that $\mathcal{R}+\mathcal{B}$ is a simple point process that is almost surely non-equidistant and has no descending chains. Then there exists an almost surely unique

two-colour stable matching scheme, which is an isometry-equivariant factor and may be constructed by iteratively matching mutually-closest $\mathcal{R}$ / $\mathcal{B}$ pairs.

Homogeneous Poisson process are non-equidistant and have no descending chains (see [7]). Descending chains are investigated in detail in [2], where it is shown in particular that they are absent in many well-studied point processes.

In this paper, our interest in stable matching lies in the typical distance between matched pairs. Let $\mathcal{M}$ be the one-colour stable matching scheme for $\mathcal{R}$. Consider the distribution function

$$F(r) := \left(\mathbb{E}\mathcal{R}([0,1)^d)\right)^{-1}\mathbb{E}\#\left\{x \in [\mathcal{R}] \cap [0,1)^d : \|x - \mathcal{M}(x)\| \le r\right\}. \tag{3}$$

As in [9], let $X$ be a random variable with probability measure $\mathbb{P}^*$ and expectation operator $\mathbb{E}^*$ such that $\mathbb{P}^*(X \le r) = F(r)$ for all $r \ge 0$. One may interpret $X$ as the distance from the origin to its partner under the Palm version of $(\mathcal{R}, \mathcal{M})$ in which we condition on the presence of an $\mathcal{R}$-point at the origin; see [9] for details. For the two-colour stable matching scheme of point processes $\mathcal{R}, \mathcal{B}$ we define $X$, $\mathbb{P}^*$, and $\mathbb{E}^*$ in the same way.

**Theorem 7** (One-colour stable matching)**.** *Let $\mathcal{R}$ be a translation-invariant ergodic point process on $\mathbb{R}^d$ with finite intensity that almost surely is non-equidistant and has no descending chains. If $\mathcal{R}$ is insertion-tolerant or deletion-tolerant, then the one-colour stable matching scheme satisfies $\mathbb{E}^* X^d = \infty$.*

**Theorem 8** (Two-colour stable matching)**.** *Let $\mathcal{R}$ and $\mathcal{B}$ be independent translation-invariant ergodic point processes on $\mathbb{R}^d$ with equal finite intensity such that the point process $\mathcal{R} + \mathcal{B}$ is non-equidistant and has no descending chains. If $\mathcal{R}$ or $\mathcal{B}$ is deletion-tolerant or insertion-tolerant, then the two-colour stable matching scheme satisfies $\mathbb{E}^* X^d = \infty$.*

Theorems 7 and 8 strengthen the earlier results in [9] in the following ways. In [9], Theorem 7 is proved in the case of homogeneous Poisson processes, but the same proof is valid under the condition that $\mathcal{R}$ is *both* insertion-tolerant *and* deletion-tolerant. Similarly, in [9], Theorem 8 is proved in the Poisson case, but the proof applies whenever $\mathcal{R}$ or $\mathcal{B}$ is insertion-tolerant. Related results appear also in [8, Theorems 32,33]

The following complementary bound is proved in [9] for Poisson processes, but again the proof given there applies more generally as follows.

**Theorem 9** ([9, Theorem 5]). *Let $\mathcal{R}$ be a translation-invariant ergodic non-equidistant point process on $\mathbb{R}^d$ with no descending chains, and unit intensity. The one-colour stable matching scheme satisfies $\mathbb{P}^*(X > r) \leq Cr^{-d}$ for all $r > 0$, for some constant $C = C(d)$ that does not depend on $\mathcal{R}$.*

Thus, Theorems 7 and 9 provide strikingly close upper and lower bounds on $X$ for the one-colour stable matching schemes of a wide range of point processes. For two-colour stable matching, even in the case of two independent Poisson processes, the correct power law for the tail of $X$ is unknown in dimensions $d \geq 2$; for $d = 1$ the bounds $\mathbb{E}^* X^{\frac{1}{2}} = \infty$ and $\mathbb{P}^*(X > r) \leq Cr^{-1/2}$ hold. See [9] for details.

The rest of the paper is organized as follows. In Section 2 we present examples. In Section 3 we prove some of the simpler results including Theorems 1 and 2. Despite the similarities between insertion-tolerance and deletion-tolerance, the proof of Theorem 2 relies on the following natural lemma, whose analogue for deletion-tolerance is false (see Example 5).

**Lemma 10** (Monotonicity of insertion-tolerance). *Let $\Pi$ be a point process on $\mathbb{R}^d$ and let $S \in \mathfrak{B}$ have finite nonzero Lebesgue measure. If $\Pi$ is insertion-tolerant, and $U$ is uniformly distributed in $S$ and independent of $\Pi$, then $\Pi + \delta_U$ is insertion-tolerant.*

Section 4 deals with Theorems 4 and 5. In Sections 5 and 6 we prove the results concerning continuum percolation and stable matchings. Section 7 provides proofs relating to some of the more elaborate examples from Section 2.

## 2. Examples

First, we give examples of (translation-invariant) point processes that possess various combinations of insertion-tolerance and deletion-tolerance. We also provide examples to show that certain results concerning insertion-tolerance do not have obvious analogues in the setting of deletion-tolerance. Second, we give examples to show that the conditions in the results concerning continuum percolation and stable matching are needed. Finally, we provide results on perturbed lattice processes and Gaussian zeros processes on the Euclidean and hyperbolic planes.

### 2.1. **Elementary examples.**

**Example 1** (Poisson process). The homogeneous Poisson point process $\Pi$ on $\mathbb{R}^d$ is both insertion-tolerant and deletion-tolerant. This follows

immediately from Theorem 4 (ii) and Theorem 5 and the relation

$$\Pi^* \stackrel{d}{=} \Pi + \delta_0.$$

It is also easy to give an direct proof of insertion-tolerance and to prove deletion-tolerance via Theorem 1 (iii). $\diamondsuit$

For $S \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, write $x + S := \{x + z : z \in S\}$.

**Example 2** (Randomly shifted lattice)**.** Let $U$ be uniformly distributed in $[0,1]^d$. Consider the point process given by $[\Lambda] := U + \mathbb{Z}^d$. Clearly, $\Lambda$ is translation-invariant. Since no ball of radius $1/4$ can contain more than one $\Lambda$-point, by Theorem 2 (ii), $\Lambda$ is not insertion-tolerant. Also the cube $[0,1]^d$ must contain $\Lambda$-points, so by Theorem 1 (iii), $\Lambda$ is not deletion-tolerant. $\diamondsuit$

**Example 3** (Randomly shifted site percolation)**.** Let $\{Y_z\}_{z\in\mathbb{Z}^d}$ be i.i.d. $\{0,1\}$-valued random variables with $\mathbb{E}Y_0 = p \in (0,1)$. Consider the random set $W := \big\{z \in \mathbb{Z}^d : Y_z = 1\big\}$. Let $U$ be uniformly distributed in $[0,1]^d$ and independent of $W$. From Theorem 1 (iii), it is easy to see that $\Lambda$ given by $[\Lambda] := U + W$ is deletion-tolerant. Clearly, as in Example 2, $\Lambda$ is not insertion-tolerant. Moreover, it is easy to verify that almost surely $[\Lambda] \cap \mathbb{Z}^d = \emptyset$, but $[\Lambda^*] \subset \mathbb{Z}^d$. Thus (1) is not satisfied. $\diamondsuit$

**Example 4** (Superposition of a Poisson point process with a randomly shifted lattice)**.** Let $\Pi$ be a Poisson point process on $\mathbb{R}^d$ and let $\Lambda$ be a randomly shifted lattice (as in Example 2) that is independent of $\Pi$. Consider the point process $\Gamma := \Pi + \Lambda$. The insertion-tolerance of $\Pi$ is inherited by $\Gamma$, but $\Gamma$ is no longer deletion-tolerant. As in Example 2, $[0,1]^d$ must contain $\Gamma$-points. $\diamondsuit$

**Example 5** (Non-monotonicity of deletion-tolerance)**.** We show that in contrast with Lemma 10, deleting a point from a deletion-tolerant process may destroy deletion-tolerance. Let $(N_i)_{i\in\mathbb{Z}}$ be i.i.d., taking values $0,1,2$ each with probability $1/3$, and let $\Pi$ have exactly $N_i$ points in the interval $[i, i+1)$, for each $i \in \mathbb{Z}$, with their locations chosen independently and uniformly at random in the interval. It is easy to verify that $\Pi$ is deletion-tolerant using Theorem 1 (iii).

Consider the $\Pi$-point $Z$ defined as follows. If the first integer interval $[i, i+1)$ to the right of the origin that contains at least one $\Pi$-point contains exactly two $\Pi$-points, then let $Z$ be the point in this interval that is closest to the origin; otherwise, let $Z$ be the closest $\Pi$-point to the left of the origin. The point process $\Pi' = \Pi - \delta_Z$ has the property that the first interval to the right of the origin that contains any $\Pi$-points contains exactly one $\Pi$-point.

Let $Z'$ be the first $\Pi'$-point to the right of the origin. The process $\Pi'' := \Pi' - \delta_{Z'}$ has the property that with non-zero probability the first interval to the right of the origin that contains any $\Pi''$-points contains exactly two $\Pi''$-points. Thus $\Pi'$ is not deletion-tolerant.

If desired, the above example can be made translation-invariant by applying a random shift $U$ as before. $\Diamond$

**Example 6** (One set $S$ satisfying $\Pi|_{S^c} \prec \Pi$ does not suffice for deletion-tolerance)**.** Let $\Lambda$ be a randomly shifted lattice in $d = 1$ (as in Example 2) and let $\Pi$ be a Poisson point process on $\mathbb{R}$ of intensity 1 that is independent of $\Lambda$. Let $Y := \cup_{x \in [\Pi]} B(x, 5)$, and consider $\Gamma := \Lambda|_{Y^c}$. Let $Z$ be the first $\Gamma$-point to the right of the origin such that $Z + i \in [\Gamma]$ for all integers $i$ with $|i| \leq 20$. Clearly, $\Gamma - \delta_Z \not\prec \Gamma$ and thus $\Gamma$ is not deletion-tolerant. On the other hand, since $\Pi$ is insertion-tolerant, $\Gamma|_{B(0,5)^c} \prec \Gamma$. (Note the contrast with Proposition 3 for insertion-tolerance.) $\Diamond$

### 2.2. **Continuum percolation and stable matching.**

**Example 7** (A point process that is neither insertion-tolerant nor deletion-tolerant and has infinitely many unbounded clusters)**.** Let $\{Y_z\}_{z\in\mathbb{Z}}$ be i.i.d. $\{0, 1\}$-valued random variables with $\mathbb{E}Y_0 = \frac{1}{2}$. Let

$$W := \left\{(x_1, x_2) \in \mathbb{Z}^2 : Y_{x_2} = 1\right\}$$

and let $U$ be uniformly distributed in $[0, 1]^2$ and independent of $W$. Consider the point process $\Lambda$ with support $U + W$. Thus $\Lambda$ is a randomly shifted lattice with columns randomly deleted. As in Example 2, $\Lambda$ is neither insertion-tolerant nor deletion-tolerant. In the continuum percolation model with parameter $R = 2$, the occupied region $\mathcal{O}(\Lambda)$ has infinitely many unbounded clusters. $\Diamond$

**Example 8** (A point process that is not insertion-tolerant, but is deletion-tolerant and has infinitely many unbounded clusters)**.** Let $\Lambda$ be a randomly shifted super-critical site percolation in $d = 2$, as in Example 3. Let $\{\Lambda_i\}_{i\in\mathbb{Z}}$ be independent copies of $\Lambda$. Let $\{Y_z\}_{z\in\mathbb{Z}}$ be i.i.d. $\{0, 1\}$-valued random variables independent of $\Lambda$ with $\mathbb{E}Y_0 = \frac{1}{2}$. Consider the point process $\Gamma$ with support

$$[\Gamma] = \bigcup_{i\in\mathbb{Z}:\, Y_i=1} [\Lambda_i] \times \{i\} \,.$$

Thus $\Gamma$ is a point process in $\mathbb{R}^3$, obtained by stacking independent copies of $\Lambda$. Clearly, the point process $\Gamma$ is deletion-tolerant, but not insertion-tolerant. With $R = 2$, the occupied region $\mathcal{O}(\Gamma)$ has infinitely many unbounded clusters. $\Diamond$

**Example 9** (One-colour matching for two perturbed lattices)**.** Let $W = \{W_i\}_{i\in\mathbb{Z}^d}$ and $Y = \{Y_i\}_{i\in\mathbb{Z}^d}$ be all i.i.d. random variables uniformly distributed in $B(0, 1/4)$. Let $U$ be uniformly distributed in $[0, 1]^d$ and independent of $W, Y$. Let $\mathcal{R}$ be the point process with support

$$[\mathcal{R}] = U + \left\{i + W_i, i + Y_i : i \in \mathbb{Z}^d\right\}.$$

It is easy to verify that $\mathcal{R}$ is neither insertion-tolerant nor deletion-tolerant, and that $\mathcal{R}$ has no descending chains and is non-equidistant. The one-colour stable matching scheme satisfies $\|x - \mathcal{M}(x)\| < \frac{1}{2}$ for all $x \in [\mathcal{R}]$ (in contrast with the conclusion in Theorem 7). $\Diamond$

**Example 10** (Two-colour matching for randomly shifted lattices)**.** Let $\mathcal{R}$ and $\mathcal{B}$ be two independent copies of the randomly shifted lattice $\mathbb{Z}$ in $d = 1$ as defined in Example 2. Although $\mathcal{R} + \mathcal{B}$ is not non-equidistant, it is easy to verify that there is an a.s. unique two-colour stable matching scheme for $\mathcal{R}$ and $\mathcal{B}$, and it satisfies $\|x - \mathcal{M}(x)\| < \frac{1}{2}$ for all $x \in [\mathcal{R}]$. $\Diamond$

2.3. **Perturbed lattices and Gaussian zeros.** The proofs of the results stated below are given in Section 7.

**Example 11** (Perturbed lattices)**.** Let $\{Y_z\}_{z\in\mathbb{Z}^d}$ be i.i.d. $\mathbb{R}^d$-valued random variables. Consider the point process $\Lambda$ given by

$$[\Lambda] := \left\{z + Y_z : z \in \mathbb{Z}^d\right\}.$$

Note that $\Lambda$ is invariant and ergodic under shifts of $\mathbb{Z}^d$. It is easy to see that (for all dimensions $d$) if $Y_0$ has bounded support, then $\Lambda$ is neither insertion-tolerant nor deletion-tolerant. Indeed, in this case we have $\Lambda(B(0, 1)) \le M$ for some constant $M < \infty$, so, by Theorem 2 (ii), $\Lambda$ is not insertion-tolerant (otherwise we could add $M+1$ random points in $B(0, 1)$). Also, $\Lambda(B(0, N)) \ge 1$, for some $N < \infty$, so Theorem 1 (iii) shows that $\Lambda$ is not deletion-tolerant. $\Diamond$

For dimensions 1 and 2 we can say more.

**Proposition 11** (Perturbed lattices in dimensions 1, 2)**.** *Let* $[\Lambda] := \left\{z + Y_z : z \in \mathbb{Z}^d\right\}$ *for i.i.d.* $\{Y_z\}_{z\in\mathbb{Z}^d}$*. For* $d = 1, 2$*, if* $\mathbb{E}\|Y_0\|^d < \infty$*, then* $\Lambda$ *is neither insertion-tolerant nor deletion-tolerant.*

**Question 1.** *Does there exists a distribution for the perturbation* $Y_0$ *such that the resulting perturbed lattice is insertion-tolerant? In particular, in the case* $d = 1$*, does this hold whenever* $Y_0$ *has infinite mean? What are the possible combinations of insertion-tolerance and deletion-tolerance for perturbed lattices? Allan Sly has informed us that he has made progress on these questions.*

Perturbed lattice models were considered by Sodin and Tsirelson [21] as simplified models to illustrate certain properties of Gaussian zero processes (which we will discuss next). Our proof of Proposition 11 is in part motivated by their remarks, and similar proofs have also been suggested by Omer Angel and Yuval Peres (personal communications).

The Gaussian zero processes on the plane and hyperbolic planes are defined as follows (see [11, 21] for background). Let $\{a_n\}_{n=0}^{\infty}$ be i.i.d. standard complex Gaussian random variables with probability density $\pi^{-1}\exp(-|z|^2)$ with respect to Lebesgue measure on the complex plane. Firstly, consider the entire function

$$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{n!}} z^n. \tag{4}$$

The set of zeros of $f$ forms a translation-invariant point process $\Upsilon_{\mathbb{C}}$ in the complex plane. Secondly, consider the analytic function on the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ given by

$$g(z) := \sum_{n=0}^{\infty} a_n z^n. \tag{5}$$

The set of zeros of $g$ forms a point process $\Upsilon_{\mathbb{D}}$. We endow $\mathbb{D}$ with the hyperbolic metric $|dz|/(1-|z|^2)$ and the group of symmetries $G$ given by the maps $z \mapsto (az+b)/(\bar{b}z+\bar{a})$, where $a, b \in \mathbb{C}$ and $|a|^2 - |b|^2 = 1$. Then $\Upsilon_{\mathbb{D}}$ is invariant in law under action of $G$.

The following two facts were suggested to us by Yuval Peres, and are consequences of results of [21] and [20] respectively.

**Proposition 12.** *The Gaussian zero process $\Upsilon_{\mathbb{C}}$ on the plane is neither insertion-tolerant nor deletion-tolerant.*

**Proposition 13.** *The Gaussian zero process $\Upsilon_{\mathbb{D}}$ on the hyperbolic plane is both insertion-tolerant and deletion-tolerant.*

## 3. Basic results

In this section we prove elementary results concerning insertion-tolerance and deletion-tolerance. The following simple application of Fubini's theorem will be useful. Recall that $\mathcal{L}$ denotes Lebesgue measure.

**Remark 2.** *Let $\Pi$ be a point process on $\mathbb{R}^d$. If $S \in \mathfrak{B}$ is a set of positive finite measure and $U$ is uniformly distributed $S$ and independent of $\Pi$, then*

$$\mathbb{P}(\Pi + \delta_U \in \cdot) = \frac{1}{\mathcal{L}(S)} \int_S \mathbb{P}(\Pi + \delta_x \in \cdot)\, dx. \qquad \Diamond$$

Let $\mathfrak{M}$ be the product $\sigma$-field on $\mathbb{M}$. For $\mathcal{A} \in \mathfrak{M}$ and $x \in \mathbb{R}^d$, we set

$$\mathcal{A}^x := \{\mu \in \mathbb{M} : \mu + \delta_x \in \mathcal{A}\}.$$

Thus $\mathcal{A}^x$ is the set of point measures for which adding a point at $x$ results in an element of $\mathcal{A}$.

*Proof of Lemma 10.* Let $\Pi$ be insertion-tolerant. We first show that for almost all $x \in \mathbb{R}^d$ the point process $\Pi + \delta_x$ is insertion-tolerant. The proof follows from the definition of $\mathcal{A}^x$. Let $V$ be uniformly distributed in $S' \in \mathfrak{B}$ and independent of $\Pi$. Suppose $\mathcal{A} \in \mathfrak{M}$ is such that

$$\mathbb{P}(\Pi + \delta_x + \delta_V \in \mathcal{A}) = \mathbb{P}(\Pi + \delta_V \in \mathcal{A}^x) > 0.$$

Since $\Pi$ is insertion-tolerant, $0 < \mathbb{P}(\Pi \in \mathcal{A}^x) = \mathbb{P}(\Pi + \delta_x \in \mathcal{A})$.

Next, let $U$ be uniformly distributed in $S \in \mathfrak{B}$ and independent of $(\Pi, V)$. Let $\mathbb{P}(\Pi + \delta_U \in \mathcal{A}) = 0$, for some $\mathcal{A} \in \mathfrak{M}$. By Remark 2, $\mathbb{P}(\Pi + \delta_x \in \mathcal{A}) = 0$ for almost all $x \in S$, and since $\Pi + \delta_x$ is insertion-tolerant for almost all $x \in \mathbb{R}^d$, we deduce that $\mathbb{P}(\Pi + \delta_x + \delta_V \in \mathcal{A}) = 0$ for almost all $x \in S$. Applying Remark 2 to the process $\Pi + \delta_V$, we obtain $\mathbb{P}(\Pi + \delta_U + \delta_V \in \mathcal{A}) = 0$. □

With Lemma 10 we prove that insertion-tolerance implies the following stronger variant of Theorem 2 in which we allow the number of points added to be random. If $(X_1, \ldots, X_n)$ is a random vector in $(\mathbb{R}^d)^n$ with law that is absolutely continuous with respect to Lebesgue measure, then we say that the random (unordered) set $\{X_1, \ldots, X_n\}$ is **nice**. A finite point process $\mathcal{F}$ is **nice** if for all $n \in \mathbb{N}$, conditional on $\mathcal{F}(\mathbb{R}^d) = n$, the support $[\mathcal{F}]$ is equal in distribution to some nice random set; we also say that the law of $\mathcal{F}$ is nice if $\mathcal{F}$ is nice.

**Corollary 14.** *Let $\Pi$ be an insertion-tolerant point process on $\mathbb{R}^d$ and let $\mathcal{F}$ be a finite point process on $\mathbb{R}^d$. If $\mathcal{F}$ admits a conditional law given $\Pi$ that is nice, then $\Pi + \mathcal{F} \prec \Pi$.*

*Proof of Theorem 2.* Clearly, (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). From Corollary 14, it is immediate that (i) $\Rightarrow$ (iii). □

*Proof of Corollary 14.* Let $U$ be uniformly distributed in $[0, 1]$ and independent of $\Pi$. Let $f : \mathbb{M} \times [0, 1] \to \mathbb{M}$ be a measurable function such that for all $\pi \in \mathbb{M}$ we have that $f(\pi, U)$ is a nice finite point process. It suffices to show that $\Pi + f(\Pi, U) \prec \Pi$.

Consider the events

$$E_{n,k} := \Big\{ f(\Pi, U)(\mathbb{R}^d) = n \Big\} \ \cap \ \Big\{ [f(\Pi, U)] \subset B(0, k) \Big\}.$$

Let $\{U_{r,k}\}_{r=1}^n$ i.i.d. random variables uniformly distributed in $B(0, k)$ and independent of $(\Pi, U)$. Let $\mathcal{F}'_{n,k} := \sum_{r=1}^n \delta_{U_{r,k}}$. By applying

Lemma 10, $n$ times, we see that $\Pi + \mathcal{F}'_{n,k} \prec \Pi$; thus it suffices to show that $\Pi + f(\Pi, U) \prec \Pi + \mathcal{F}'_{n,k}$ for some $n, k \geq 0$.

For each $\mathbf{x} \in (\mathbb{R}^d)^n$, let $(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathbf{x}$. If $S \subset \mathbb{R}^d$ has $n$ elements, then we write $\langle S \rangle := (s_1 \ldots, s_n) \in (\mathbb{R}^d)^n$, where $s_i$ are the elements of $S$ in lexicographic order. For each $n \geq 0$, let $g_n : (\mathbb{R}^d)^n \times \mathbb{M} \to \mathbb{R}$ be a measurable function such that $g_n(\cdot, \pi)$ is the probability density function (with respect to $n$-dimensional Lebesgue measure) of $\langle [f(\pi, U)] \rangle$, conditional on $f(\pi, U)(\mathbb{R}^d) = n$. Let $Q$ be the law of $\Pi$ and let $\mathcal{A} \in \mathfrak{M}$. Thus

$$\mathbb{P}\big(\Pi + f(\Pi, U) \in \mathcal{A},\ E_{n,k}\big) = \\ \int \bigg( \int_{B(0,k)^n} \mathbf{1}\Big[\pi + \sum_{i=1}^n \delta_{\mathbf{x}_i} \ \in \mathcal{A}\Big] g(\mathbf{x}, \pi) d\mathbf{x} \bigg) dQ(\pi). \tag{6}$$

On the other hand,

$$\mathbb{P}\big(\Pi + \mathcal{F}'_{n,k} \in \mathcal{A}\big) = \\ \int \frac{1}{\mathcal{L}(B(0,k))^n} \bigg( \int_{B(0,k)^n} \mathbf{1}\Big[\pi + \sum_{i=1}^n \delta_{\mathbf{x}_i} \ \in \mathcal{A}\Big] d\mathbf{x} \bigg) dQ(\pi). \tag{7}$$

If $\mathbb{P}(\Pi + f(\Pi, U) \in \mathcal{A}) > 0$, then there exist $n, k \geq 0$ such that $\mathbb{P}(\Pi + f(\Pi, U) \in \mathcal{A},\ E_{n,k}) > 0$; moreover from (6) and (7), we deduce that $\mathbb{P}(\Pi + \mathcal{F}'_{n,k} \in \mathcal{A}) > 0$. $\square$

The proof of Theorem 1 relies on the following lemma.

**Lemma 15.** *Let $\Pi$ be a point process on $\mathbb{R}^d$. If $\mathcal{F}$ is a finite subprocess of $\Pi$, then there exists $S \in \mathfrak{B}$ with $\mathcal{L}(S) \in (0, \infty)$ such that*

$$\mathbb{P}(\Pi|_S = \mathcal{F}) > 0. \tag{8}$$

*Proof.* A ball $B(x, r)$ is **rational** if $x \in \mathbb{Q}^d$ and $r \in \mathbb{Q}^+$. Let $C$ be the collection of all unions of finitely many rational balls. Clearly $C$ is countable. We will show that there exists $S \in C$ satisfying (8). Since $\Pi$ is locally finite, it follows that there exists a $C$-valued random variable $\mathbf{S}$ such that $\Pi|_{\mathbf{S}} = \mathcal{F}$ a.s. Since

$$\sum_{S \in C} \mathbb{P}(\Pi|_S = \mathcal{F}, \ \ \mathbf{S} = S) = \mathbb{P}(\Pi|_{\mathbf{S}} = \mathcal{F}) = 1,$$

at least one of the terms of the sum is nonzero. $\square$

With Lemma 15 we first prove the following special case of Theorem 1.

**Lemma 16.** *Let $\Pi$ be a point process on $\mathbb{R}^d$. The following conditions are equivalent.*

*(i) The point process $\Pi$ is deletion-tolerant.*
*(ii) If $\mathcal{F}$ is a finite subprocess of $\Pi$ such that $\mathcal{F}(\mathbb{R}^d)$ is a bounded random variable, then $\Pi - \mathcal{F} \prec \Pi$.*

*Proof.* Clearly, (ii) implies (i).

We show by induction on the number of points of the finite subprocess that (i) implies (ii). Assume that $\Pi$ is deletion-tolerant. Suppose that (ii) holds for every finite subprocess $\mathcal{F}$ of $\Pi$ such that $\mathcal{F}(\mathbb{R}^d) \leq n$. Let $\mathcal{F}'$ be a finite subprocess of $\Pi$ with $\mathcal{F}'(\mathbb{R}^d) \leq n+1$. Observe that on the event that $\mathcal{F}'(\mathbb{R}^d) \neq 0$, we have $\mathcal{F}' = \mathcal{F} + \delta_Z$, where $\mathcal{F}$ is a finite subprocess of $\Pi$ with $\mathcal{F}(\mathbb{R}^d) \leq n$ and $Z$ is some $\Pi$-point. Let $\mathbb{P}(\Pi - \mathcal{F}' \in \mathcal{A}) > 0$, for some $\mathcal{A} \in \mathfrak{M}$. If $\mathbb{P}(\Pi - \mathcal{F}' \in \mathcal{A},\ \ \mathcal{F}'(\mathbb{R}^d) = 0) > 0$, then clearly $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. Thus we assume without loss of generality that $\mathcal{F}' = \mathcal{F} + \delta_Z$ so that $\mathbb{P}(\Pi - \mathcal{F} - \delta_Z \in \mathcal{A}) > 0$. By applying Lemma 15 to the point process $\Pi - \mathcal{F}$, conditioned on $\Pi - \mathcal{F} - \delta_Z \in \mathcal{A}$, there exists $S \in \mathfrak{B}$ with finite Lebesgue measure, so that

$$\mathbb{P}\Big((\Pi - \mathcal{F})|_S = \delta_Z \ \Big|\ \Pi - \mathcal{F} - \delta_Z \in \mathcal{A}\Big) > 0. \tag{9}$$

Let $\mathcal{A}^S := \{\mu + \delta_x : \mu \in \mathcal{A},\ x \in S\}$, so that by the definition of $\mathcal{A}^S$ and (9), we have $\mathbb{P}(\Pi - \mathcal{F} \in \mathcal{A}^S) > 0$. By the inductive hypothesis, $\mathbb{P}(\Pi \in \mathcal{A}^S) > 0$.

Observe that if $\Pi \in \mathcal{A}^S$, there is an $x \in [\Pi] \cap S$ such that $\Pi - \delta_x \in \mathcal{A}$. Define a $\Pi$-point $R$ as follows. If $\Pi \in \mathcal{A}^S$, let $R$ be the point of $[\Pi] \cap S$ closest to the origin (where ties are broken using lexicographic order) such that $\Pi - \delta_R \in \mathcal{A}$, otherwise let $R$ be the $\Pi$-point closest to the origin. Hence

$$\mathbb{P}(\Pi - \delta_R \in \mathcal{A}) \geq \mathbb{P}(\Pi \in \mathcal{A}^S) > 0.$$

Since $\Pi$ is deletion-tolerant, $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. ☐

*Proof of Theorem 1.* We show that (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Assume that (iii) holds and that for some $\Pi$-point $Z$ and some $\mathcal{A} \in \mathfrak{M}$ we have $\mathbb{P}(\Pi - \delta_Z \in \mathcal{A}) > 0$. By Lemma 15, $\mathbb{P}(\Pi|_{S^c} \in \mathcal{A}) > 0$ for some $S \in \mathfrak{B}$, with finite Lebesgue measure. From (iii), $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. Thus (i) holds and $\Pi$ is deletion-tolerant.

Assume that (i) holds. Let $\mathcal{F}$ be a finite subprocess of $\Pi$ and suppose for some $\mathcal{A} \in \mathfrak{M}$ we have $\mathbb{P}(\Pi - \mathcal{F} \in \mathcal{A}) > 0$. Define $\mathcal{F}_n$ as follows. Take $\mathcal{F}_n = \mathcal{F}$ if $\mathcal{F}(\mathbb{R}^d) = n$, otherwise set $\mathcal{F}_n = 0$. Note that for some $n$, we have $\mathbb{P}(\Pi - \mathcal{F}_n \in \mathcal{A}) > 0$. Since $\Pi$ is deletion-tolerant, by Lemma 16, $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. Thus (ii) holds.

Clearly (ii) implies (iii), since for any set $S \in \mathfrak{B}$ with finite measure, the point process with support $[\Pi] \cap S$ is a finite subprocess of $\Pi$. $\square$

For a translation $\theta$ of $\mathbb{R}^d$ and a point measure $\mu \in \mathbb{M}$, we define $\theta\mu \in \mathbb{M}$ by $(\theta\mu)(S) := \mu(\theta^{-1}S)$ for all $S \in \mathfrak{B}$; for $\mathcal{A} \in \mathfrak{M}$, we write $\theta\mathcal{A} := \{\theta\mu : \mu \in \mathcal{A}\}$. For $x \in \mathbb{R}^d$ let $\theta_x$ be the translation defined by $\theta_x(y) := y + x$ for all $y \in \mathbb{R}^d$.

*Proof of Proposition 3.* Let $U, V$ be uniformly distributed on $S, T \in \mathfrak{B}$ respectively and let $U, V, \Pi$ be independent. Assume that $\Pi + \delta_U \prec \Pi$ and let $\mathcal{A} \in \mathfrak{M}$ be such that $\mathbb{P}(\Pi + \delta_V \in \mathcal{A}) > 0$. We will show that $\mathbb{P}(\Pi \in \mathcal{A}) > 0$.

Since $\Pi$ is translation-invariant, for all $\mathcal{A}' \in \mathfrak{M}$ we have $\mathbb{P}(\Pi + \delta_{\theta U} \in \mathcal{A}') = \mathbb{P}(\Pi + \delta_U \in \theta^{-1}\mathcal{A}')$ and thus $\Pi + \delta_{\theta U} \prec \Pi$ for all translations $\theta$ of $\mathbb{R}^d$. By Remark 2, $T' := \{w \in T : \mathbb{P}(\Pi + \delta_w \in \mathcal{A}) > 0\}$ has positive Lebesgue measure. By the Lebesgue density theorem [18, Corollary 2.14], there exist $x \in T'$, $y \in S$, and $\varepsilon > 0$ such that

$$\begin{aligned} \mathcal{L}(T \cap B(x, \varepsilon)) &> \tfrac{1}{2}\mathcal{L}B(x, \varepsilon); \\ \mathcal{L}(S \cap B(y, \varepsilon)) &> \tfrac{1}{2}\mathcal{L}B(y, \varepsilon). \end{aligned}$$

Thus with $z = x - y$, the set $T' \cap \theta_z S$ has positive Lebesgue measure. Thus by Remark 2, $\mathbb{P}(\Pi + \delta_{\theta_z U} \in \mathcal{A}) > 0$. Since $\Pi + \delta_{\theta_z U} \prec \Pi$, we have $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. $\square$

## 4. Palm equivalences

In this section, we discuss insertion-tolerance and deletion-tolerance in the context of Palm processes. We begin by presenting some standard definitions and facts. Let $\Pi$ be a translation-invariant point process with finite intensity $\lambda$. The Palm version of $\Pi$ is the point process $\Pi^*$ such that for all $\mathcal{A} \in \mathfrak{M}$ and all $S \in \mathfrak{B}$ with finite Lebesgue measure, we have

$$\mathbb{E}\#\big\{x \in [\Pi] \cap S \text{ with } \Pi \in \theta_x\mathcal{A}\big\} = \lambda\mathcal{L}S \cdot \mathbb{P}(\Pi^* \in \mathcal{A}), \tag{10}$$

where $\#B$ denotes the cardinality of a set $B$. Sometimes (10) is called the *Palm property*.

By a monotone class argument, a consequence of (10) is that for all measurable $f : \mathbb{M} \times \mathbb{R}^d \to [0, \infty)$ we have

$$\mathbb{E}\int_{\mathbb{R}^d} f(\theta_{-x}\Pi, x)\, d\Pi(x) = \lambda \int_{\mathbb{R}^d} \mathbb{E}f(\Pi^*, x)\, dx; \tag{11}$$

see [13, Chapter 11].

*Proof of Theorem 5.* Let $\Pi$ have intensity $\lambda > 0$. Let $S \in \mathfrak{B}$ have finite Lebesgue measure. By Theorem 1 it suffices to show that $\Pi|_{S^c} \prec \Pi$.

Let $\mathbb{P}(\Pi|_{S^c} \in \mathcal{A}) > 0$, for some $\mathcal{A} \in \mathfrak{M}$. Thus we may assume that

$$\mathbb{P}(\exists x \in [\Pi] \cap S : \Pi - \delta_x \in \mathcal{A}) > 0, \tag{12}$$

otherwise $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. By applying (11) to the function

$$(\mu, x) \mapsto \mathbf{1}[\mu - \delta_0 \in \theta_{-x}\mathcal{A}]\mathbf{1}[x \in S],$$

we obtain

$$\mathbb{E}\#\left\{x \in [\Pi] \cap S : \theta_{-x}(\Pi - \delta_x) \in \theta_{-x}\mathcal{A}\right\} = \lambda \int_S \mathbb{P}(\Pi^* - \delta_0 \in \theta_{-x}\mathcal{A}) dx. \tag{13}$$

From (12) and (13), we deduce that $\mathbb{P}(\Pi^* - \delta_0 \in \theta_{-x}\mathcal{A}) > 0$, for some $x \in S$. By assumption, $\mathbb{P}(\Pi \in \theta_{-x}\mathcal{A}) > 0$. Since $\Pi$ is translation-invariant, $\mathbb{P}(\Pi \in \mathcal{A}) > 0$. $\square$

*Proof of Theorem 4, (i) $\Rightarrow$ (ii).* Suppose that $\Pi + \delta_0$ is not absolutely continuous with respect to $\Pi^*$; then there exists $\mathcal{A} \in \mathfrak{M}$ such that

$$\mathbb{P}(\Pi^* \in \mathcal{A}) = 0 \quad \text{but} \quad \mathbb{P}(\Pi + \delta_0 \in \mathcal{A}) > 0.$$

Without loss of generality, take $\mathcal{A}$ to be a set that does not care whether there is a point at 0; that is if $\mu \in \mathcal{A}$, then $\mu' \in \mathcal{A}$, provided $\mu, \mu'$ agree on $\mathbb{R}^d \setminus \{0\}$. By translation-invariance,

$$0 < c := \mathbb{P}(\Pi + \delta_0 \in \mathcal{A}) = \mathbb{P}(\Pi \in \mathcal{A}) = \mathbb{P}(\Pi \in \theta_x\mathcal{A})$$

for every $x \in \mathbb{R}^d$. Hence the translation-invariant random set $G := \{x \in \mathbb{R}^d : \Pi \in \theta_x\mathcal{A}\}$ has intensity $\mathbb{E}\mathcal{L}([0,1]^d \cap G) = c$. Moreover, if $U$ is uniformly distributed in $[0,1]^d$ and independent of $\Pi$, then $\mathbb{P}(U \in G) = c$. Therefore defining the set

$$\mathcal{A}' := \{\mu \in \mathbb{M} : \exists x \in [\mu] \cap [0,1]^d \text{ with } \mu \in \theta_x\mathcal{A}\},$$

we deduce that $\mathbb{P}(\Pi + \delta_U \in \mathcal{A}') > 0$. (Recall that $\mathcal{A}$ does not care whether there is a point at 0.) On the other hand by the Palm property (10) we have

$$\begin{aligned} \mathbb{P}(\Pi \in \mathcal{A}') &\leq \mathbb{E}\#\{x \in [\Pi] \cap [0,1]^d \text{ with } \Pi \in \theta_x\mathcal{A}\} \\ &= \lambda\mathcal{L}S \cdot \mathbb{P}(\Pi^* \in \mathcal{A}) = 0. \end{aligned}$$

Thus $\Pi$ is not insertion-tolerant. $\square$

The following observations will be useful in the proof that (ii) implies (i) in Theorem 4.

**Lemma 17.** *Let $\Pi$ be a translation-invariant point process on $\mathbb{R}^d$ with finite intensity. If $Y$ is any $\mathbb{R}^d$-valued random variable, and $U$ is uniformly distributed in $S \in \mathfrak{B}$ and independent of $(\Pi, Y)$, then $\theta_U \theta_Y \Pi \prec \Pi$.*

**Lemma 18.** *Let $\Pi$ be a translation-invariant point process on $\mathbb{R}^d$ with finite intensity. There exists a $\Pi$-point $Z$ such that $\Pi^* \prec \theta_{-Z}\Pi$.*

*Proof of Theorem 4, (ii) $\Rightarrow$ (i).* Suppose that $\Pi + \delta_0 \prec \Pi^*$. Without loss of generality we may assume that $\Pi$ and $\Pi^*$ are defined a common probability space. By Lemma 18, there exists a $\Pi$-point $Z$ such that

$$\Pi^* \prec \theta_{-Z}\Pi. \tag{14}$$

Let $U$ be uniformly distributed in a Borel set $S$ and independent of $(\Pi, \Pi^*, Z)$. By Lemma 17, it suffices to show that $\Pi + \delta_U \prec \theta_U \theta_{-Z}\Pi$. Since $U$ is independent of $(\Pi, \Pi^*, Z)$, from (14) it follows that $\theta_U \Pi^* \prec \theta_U \theta_{-Z}\Pi$. Thus it remains to show that $\Pi + \delta_U \prec \theta_U \Pi^*$.

Since $\Pi$ is translation-invariant and $U$ is independent of $\Pi$ we have

$$\theta_U(\Pi + \delta_0) \stackrel{d}{=} \Pi + \delta_U. \tag{15}$$

Since we assume that $\Pi + \delta_0 \prec \Pi^*$ and $U$ is independent of $(\Pi, \Pi^*)$ we deduce from (15) that $\Pi + \delta_U \prec \theta_U \Pi^*$. □

*Proof of Lemma 17.* Let $Q$ be the joint law of $\Pi$ and $Y$. Since $U$ is independent of $(\Pi, Y)$, by Fubini's theorem, for all $\mathcal{A} \in \mathfrak{M}$, we have

$$\begin{aligned}
\mathbb{P}(\theta_U \theta_Y \Pi \in \mathcal{A}) &= \frac{1}{\mathcal{L}(S)} \int \left( \int_S \mathbf{1}[\theta_{u+y}\pi \in \mathcal{A}] du \right) dQ(\pi, y) \\
&\leq \frac{1}{\mathcal{L}(S)} \int \left( \int_{\mathbb{R}^d} \mathbf{1}[\theta_x \pi \in \mathcal{A}] dx \right) dQ(\pi, y) \\
&= \frac{1}{\mathcal{L}(S)} \int_{\mathbb{R}^d} \mathbb{P}(\theta_x \Pi \in \mathcal{A}) dx \\
&= \frac{1}{\mathcal{L}(S)} \int_{\mathbb{R}^d} \mathbb{P}(\Pi \in \mathcal{A}) dx.
\end{aligned}$$
□

Lemma 18 is an immediate consequence of a result of Thorisson [22], which states that there exists a *shift-coupling* of $\Pi$ and $\Pi^*$; that is, a $\Pi$-point $Z$ such that $\Pi^* \stackrel{d}{=} \theta_{-Z}\Pi$. In fact, Holroyd and Peres [10] prove that such a $Z$ may be chosen as a deterministic function of $\Pi$. Since Lemma 18 is much weaker result, we can give the following simple self-contained proof.

*Proof of Lemma 18.* Let $\{a_i\}_{i \in \mathbb{N}} = [\Pi]$ be an enumeration of the $\Pi$-points. Let $K$ be a random variable with support $\mathbb{N}$; also assume that

$K$ is independent of $(a_i)_{i\in\mathbb{N}}$. Define the $\Pi$-point $Z := a_K$. We will show that $\Pi^* \prec \theta_{-Z}\Pi$.

Let $\mathcal{A} \in \mathcal{M}$ be so that $\mathbb{P}(\Pi^* \in \mathcal{A}) > 0$. By the Palm property (10), there exists a $\Pi$-point $Z' = Z'(\mathcal{A})$ such that $\mathbb{P}(\theta_{-Z'}\Pi \in \mathcal{A}) > 0$; moreover, there exists $i \in \mathbb{N}$ such that $\mathbb{P}(\theta_{-Z'}\Pi \in \mathcal{A},\ Z' = a_i) > 0$. Since $K$ is independent of $(a_i)_{i\in\mathbb{N}}$, it follows from the definition of $Z$ that

$$\mathbb{P}(\theta_{-Z'}\Pi \in \mathcal{A},\ Z' = a_i,\ K = i,\ Z = a_i) > 0.$$

Therefore, $\mathbb{P}(\theta_{-Z}\Pi \in \mathcal{A}) > 0$. □

## 5. Continuum percolation

Theorem 6 is an immediate consequence of the following. Consider the Boolean continuum percolation model for a point process $\Pi$. Let $W$ denote the cluster of containing the origin. For $M > 0$, an ***M*-branch** is an unbounded component of $W \cap B(0, M)^c$.

**Lemma 19.** *For a translation-invariant ergodic insertion-tolerant point process, the number of unbounded clusters is a fixed constant a.s. that is zero, one, or infinity.*

**Lemma 20.** *If an insertion-tolerant point process has infinitely many unbounded clusters, then with positive probability there exists $M > 0$ so that there at least three $M$-branches.*

**Theorem 21.** *For all $M > 0$, a translation-invariant ergodic point process has at most two $M$-branches.*

For a proof of Theorem 21 see [19, Theorem 7.1].

*Proof of Theorem 6.* From Lemma 19, it suffices to show that there can not be infinitely many unbounded clusters; this follows from Theorem 21 and Lemma 20. □

For $r > 0$, let $r\mathbb{Z}^d := \{rz : z \in \mathbb{Z}^d\}$.

*Proof of Lemma 19.* Let $\Pi$ be a translation-invariant ergodic insertion-tolerant point process. Let the occupied region be given by a union of balls of radius $R > 0$. By ergodicity, if $K(\Pi)$ is the number of unbounded clusters, then $K(\Pi)$ is a fixed constant a.s. Assume that $K(\Pi) < \infty$. It suffices to show that $\mathbb{P}(K(\Pi) \leq 1) > 0$. Since $K(\Pi) < \infty$, there exists $N > 0$ so that every unbounded cluster intersects $B(0, N)$ with positive probability. Consider the finite set $S := (R/4)\mathbb{Z}^d \cap B(0, N)$. For each $x \in S$, let $U_x$ be uniformly distributed in $B(x, R)$ and assume that the $U_x$ and $\Pi$ are independent. Let $\mathcal{F} := \sum_{x\in S} \delta_{U_x}$. Since $B(0, N) \subset \cup_{x\in S} B(U_x, R)$, we have that

$\mathbb{P}(K(\Pi + \mathcal{F}) \leq 1) > 0$. By Theorem 2 (ii), $\Pi + \mathcal{F} \prec \Pi$, so that $\mathbb{P}(K(\Pi) \leq 1) > 0$. □

*Proof of Lemma 20.* The proof is similar to that of Lemma 19. Let $\Pi$ be an insertion-tolerant point process with infinitely many unbounded clusters. Let the occupied region be given by a union of balls of radius $R > 0$. Choose $N$ large enough so that at least three unbounded clusters interest $B(0, N)$ with positive probability. Define a finite point process $\mathcal{F}$ exactly as in the proof of Lemma 19. The point process $\Pi + \mathcal{F}$ has at least three $(N + R)$-branches with positive probability and Theorem 2 (ii) implies that $\Pi + \mathcal{F} \prec \Pi$. Thus $\Pi$ has at least three $(N + R)$-branches with positive probability. □

## 6. Stable matching

Theorems 7 and 8 are consequences of the following lemmas. Let $\mathcal{R}$ be a point process with a unique one-colour stable matching scheme $\mathcal{M}$. Define

$$H = H(\mathcal{R}) := \big\{x \in [\mathcal{R}] : \|x - \mathcal{M}(x)\| > \|x\| - 1\big\}. \tag{16}$$

This is the set of $\mathcal{R}$-points that would prefer some $\mathcal{R}$-point in the ball $B(0, 1)$, if one were present in the appropriate location, over their current partners. Also define $H$ by (16) for the case of two-colour stable matching.

A calculation given in [9, Proof of Theorem 5(i)] shows that, for one-colour and two-colour matchings,

$$\mathbb{E}\#H = c\,\mathbb{E}^*\big[(X + 1)^d\big]. \tag{17}$$

for some $c = c(d) \in (0, \infty)$.

**Lemma 22** (One-colour stable matching)**.** *Let $\mathcal{R}$ be a translation-invariant point process on $\mathbb{R}^d$ with finite intensity that almost surely is non-equidistant and has no descending chains. If $\mathcal{R}$ is insertion-tolerant, then $\mathbb{P}(\#H = \infty) = 1$. If $\mathcal{R}$ is deletion-tolerant, then $\mathbb{P}(\#H = \infty) > 0$.*

**Lemma 23** (Two-colour stable matching)**.** *Let $\mathcal{R}$ and $\mathcal{B}$ be independent translation-invariant ergodic point processes on $\mathbb{R}^d$ with equal finite intensity, such that the point process $\mathcal{R}+\mathcal{B}$ is non-equidistant and has no descending chains. If $\mathcal{R}$ is insertion-tolerant, then $\mathbb{P}(\#H = \infty) = 1$. If $\mathcal{R}$ is deletion-tolerant, then $\mathbb{P}(\#H = \infty) > 0$.*

**Remark 3.** *Recall that in the case of two-colour stable matching we defined $X$ in terms of the distance from an $\mathcal{R}$-point to its partner. If we instead define $X'$ by replacing $\mathcal{R}$ with $\mathcal{B}$ in* (3)*, then $X' \stackrel{d}{=} X$; see the discussion after* [9, Proposition 7] *for details.* $\Diamond$

*Proof of Theorem 7.* Use Lemma 22 together with (17). $\square$

*Proof of Theorem 8.* Use Lemma 23 together with (17) and Remark 3. $\square$

The following lemmas concerning stable matchings in a deterministic setting will be needed. A **partial matching** of a point measure $\mu \in \mathbb{M}$ is the edge set $m$ of simple graph $([\mu], m)$ in which every vertex has degree at most one. A partial matching is a **perfect** matching if every vertex has degree exactly one. We write $m(x) = y$ if and only if $\{x, y\} \in m$, and set $m(x) = \infty$ if $x$ is unmatched. We say a partial matching is **stable** if there do not exist distinct points $x, y \in [\mu]$ satisfying

$$\|x - y\| < \min \{\|x - m(x)\|, \|y - m(y)\|\}, \tag{18}$$

where $\|x - m(x)\| = \infty$ if $x$ is unmatched. Note that in any stable partial matching there can be at most one unmatched point.

For each $\varepsilon > 0$, set

$$H_\varepsilon = H_\varepsilon(\mu) := \{x \in [\mu] : \|x - m(x)\| > \|x\| - \varepsilon\}.$$

For each $y \in \mathbb{R}^d$, set

$$N(\mu, y) := \{x \in [\mu] \setminus \{y\} : \|x - m(x)\| > \|x - y\|\}.$$

This is the set of $\mu$-points that would prefer $y \in \mathbb{R}^d$ over their partners.

**Lemma 24.** *If $\mu \in \mathbb{M}$ is non-equidistant and has no descending chains, then $\mu$ has an unique stable partial matching $m$. In addition, we have the following properties.*

*(i) If $\{x, y\} \in m$ is a matched pair, then $m \setminus \{\{x, y\}\}$ is the unique stable partial matching of $\mu - \delta_x - \delta_y$.*

*(ii) Let $\varepsilon > 0$. If $m$ is a perfect matching and $\#H_\varepsilon = 0$, then for all $x \in B(0, \varepsilon)$ such that $\mu + \delta_x$ is non-equidistant, $m$ is the unique stable partial matching of $\mu + \delta_x$; in particular, $x$ is unmatched in $m$.*

*(iii) If $\{x, y\} \in m$ is a matched pair and $\#N(\mu, y) = 0$, then $m \setminus \{\{x, y\}\}$ is the unique stable partial matching of $\mu - \delta_x$, and in particular, $y$ is left unmatched.*

*Proof.* The existence and uniqueness is given by [9, Lemma 15]. Thus for (i)–(iii) it suffices to check that the claimed matching is stable, which is immediate from the definition (18). $\square$

The next lemma is a simple consequence of Lemma 24.

**Lemma 25.** *Suppose that $\mu \in \mathbb{M}$ is non-equidistant and has no descending chains. Let $m$ be the unique stable matching of $\mu$. Suppose that $\{x, y\} \in m$ and $0 \notin [\mu]$. There exists $\varepsilon > 0$ such that for $\mathcal{L}$-a.a. $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$: the unique stable matching $m'$ of $\mu + \delta_{x'} + \delta_{y'}$ is given by*

$$m' = (m \setminus \{\{x, y\}\}) \cup \{\{x, x'\}, \{y, y'\}\},$$

*and furthermore, $x, x', y, y' \notin H_\varepsilon(\mu + \delta_{x'} + \delta_{y'}) \subseteq H_\varepsilon(\mu)$.*

*Proof of Lemma 25.* Consider

$$d_v := \min\{\|v - w\| : w \in [\mu] \cup \{0\},\ w \neq v\}.$$

Let

$$\varepsilon := \tfrac{1}{5} \min\{d_x, d_y, d_0\} \tag{19}$$

(any multiplicative factor less than $\frac{1}{4}$ would suffice here). Let $A := B(x, \varepsilon) \times B(y, \varepsilon)$. It is easy to verify that for $\mathcal{L}$-a.a. $(x', y') \in A$ that the measure $\mu + \delta_{x'} + \delta_{y'}$ is also non-equidistant and has no descending chains. Thus by Lemma 24, for $\mathcal{L}$-a.a. $(x', y') \in A$ the measure $\mu + \delta_{x'} + \delta_{y'}$ has a unique stable perfect matching $m'$. Clearly, by (19) and (18), we have that $\{x, x'\}, \{y, y'\} \in m'$. On the other hand, by Lemma 24 (i), $m' \setminus \{\{x, x'\}, \{y, y'\}\}$ is the unique stable perfect matching of $\mu - \delta_x - \delta_y$ and $m \setminus \{\{x, y\}\}$ is the also the unique stable perfect matching of $\mu - \delta_x - \delta_y$. Thus

$$m' = (m \setminus \{\{x, y\}\}) \cup \{\{x, x'\}, \{y, y'\}\}.$$

It also follows from (19) that

$$x, x', y, y' \notin H_\varepsilon(\mu + \delta_{x'} + \delta_{y'}) \subseteq H_\varepsilon(\mu). \qquad \square$$

*Proof of Lemma 22: the case where $\mathcal{R}$ is insertion-tolerant.* Let $\mathcal{R}$ be insertion-tolerant. Note that $H_1(\mathcal{R}) = H(\mathcal{R})$. First, we will show that

$$\mathbb{P}(\#H_\varepsilon(\mathcal{R}) > 0) = 1 \text{ for all } \varepsilon > 0. \tag{20}$$

Second, we will show that if $\mathbb{P}(0 < \#H_1(\mathcal{R}) < \infty) > 0$, then there exists a finite point process $\mathcal{F}$ such that $\mathcal{F}$ admits a nice conditional law given $\mathcal{R}$, and

$$\lim_{\varepsilon \to 0} \mathbb{P}\big(\#H_\varepsilon(\mathcal{R} + \mathcal{F}) = 0\big) = \mathbb{P}\big(0 < \#H_1(\mathcal{R}) < \infty\big) > 0. \tag{21}$$

Finally, note that by Corollary 14 and the insertion-tolerance of $\mathcal{R}$ that (21) and (20) are in contradiction. Thus $\mathbb{P}(\#H_1(\mathcal{R}) = \infty) = 1$. It remains to prove the first two assertions.

The following definition will be useful. Let $\mathbb{M}'$ be the set of point measures $\mu \in \mathbb{M}$ such that $\mu$ has a unique stable perfect matching, has no descending chains, and is non-equidistant.

Let $\varepsilon > 0$. Let $\mathcal{J}$ be the set of point measures $\mu \in \mathbb{M}'$ such that $\#H_\varepsilon(\mu) = 0$. To show (20), it suffices to prove that $\mathbb{P}(\mathcal{R} \in \mathcal{J}) = 0$. Let $\mu \in \mathcal{J}$ and let $m$ be the unique stable perfect matching for $\mu$. By Lemma 24 (ii), for Lebesgue-a.a. $x \in B(0, \varepsilon)$ the unique stable partial matching for $\mu + \delta_x$ is $m$ (and $x$ is unmatched). If $\mathbb{P}(\mathcal{R} \in \mathcal{J}) > 0$, then it follows from the insertion-tolerance of $\mathcal{R}$ that with positive probability $\mathcal{R}$ does not have a perfect stable matching, a contradiction.

Now let $\mathcal{A}$ be the set of point measures $\mu \in \mathbb{M}'$ such that $0 < \#H_1(\mu) < \infty$ and $0 \notin [\mu]$. If $\mathcal{R} \in \mathcal{A}$, then, by applying Lemma 25 repeatedly, there exists $\rho = \rho(\mathcal{R})$ such that if a point is added within distance $\rho$ of each point in $H_1(\mathcal{R})$ and each of their partners, then (for $\mathcal{L}$-a.a. choices of such points) the resulting process $\mathcal{R}'$ satisfies $H_\rho(\mathcal{R}') = 0$. Let $\mathcal{F}$ be the finite point process whose conditional law given $\mathcal{R}$ is given as follows. Take independent uniformly random points in each of the appropriate balls of radius $\rho$ provided $\mathcal{R} \in \mathcal{A}$; otherwise take $\mathcal{F} = 0$. By the construction,

$$\lim_{\varepsilon \to 0} \mathbb{P}\big(\#H_\varepsilon(\mathcal{R} + \mathcal{F}) = 0 \mid \mathcal{R} \in \mathcal{A},\ \rho(\mathcal{R}) > \varepsilon\big) = 1,$$

so (21) follows. □

*Proof of Lemma 22: the case where $\mathcal{R}$ is deletion-tolerant.* Suppose $\mathcal{R}$ is deletion-tolerant. We will show that for any $\mathcal{R}$-point $Z$

$$\#N(\mathcal{R}, Z) = \infty \text{ a.s.} \tag{22}$$

From (22) it follows that if $\mathcal{R}(B(0,1)) > 0$, then $\#H = \infty$. Since $\mathcal{R}$ is translation-invariant, $\mathbb{P}(\mathcal{R}(B(0,1)) > 0) > 0$ and $\mathbb{P}(\#H = \infty) > 0$.

It remains to show (22). Let $Z$ be an $\mathcal{R}$-point. Let $\mathcal{F}_1$ be the point process with support $N(\mathcal{R}, Z)$, and let $\mathcal{F}_2$ be the point process with support $\{\mathcal{M}(y) : y \in N(\mathcal{R}, Z)\}$. Consider the point process $\mathcal{F}$ defined by

$$\mathcal{F} \;:=\; \begin{cases} \mathcal{F}_1 + \mathcal{F}_2, & \text{if } \ \#N(\mathcal{R}, Z) < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}'$ be given by

$$[\mathcal{M}'] := [\mathcal{M}] \setminus \bigcup_{x \in [\mathcal{F}]} \{\{x, \mathcal{M}(x)\}\}.$$

By Lemma 24 (i), $\mathcal{M}'$ is the unique stable matching for $\mathcal{R} - \mathcal{F}$ a.s.

Towards a contradiction assume that $\mathbb{P}(\#N(\mathcal{R}, Z) < \infty) > 0$. Thus, $\mathbb{P}(N(\mathcal{R}-\mathcal{F}, Z) = 0) > 0$. By Lemma 24 (iii), with positive probability, $\mathcal{R} - \mathcal{F} - \delta_{\mathcal{M}(Z)}$ has the unique stable partial matching given by $\mathcal{M}'$ with the pair $\{Z, \mathcal{M}(Z)\}$ removed and $Z$ left unmatched. From Theorem 1 (ii) and the deletion-tolerance of $\mathcal{R}$ we have $\mathcal{R} - \mathcal{F} - \delta_{\mathcal{M}(Z)} \prec \mathcal{R}$. Thus with positive probability, $\mathcal{R}$ has a stable partial matching with an unmatched point, a contradiction. $\square$

We now turn to the two-colour case. Given two point measures $\mu, \mu' \in \mathbb{M}$ such that $\mu + \mu'$ is a simple point measure, we say that $m$ is a **partial** (respectively, **perfect**) matching of $(\mu, \mu')$ if $m$ is the edge set of a simple bipartite graph $([\mu], [\mu'], m)$ in which every vertex has degree at most one (respectively, exactly one). We write $m(x) = y$ if and only if $\{x, y\} \in m$ and set $m(x) = \infty$ if $x$ is unmatched. We say that $m$ is **stable** if there do not exist $x \in [\mu]$ and $y \in [\mu']$ satisfying (18). If $\mu + \mu'$ is non-equidistant and has no descending chains then there exists a unique stable partial matching of $(\mu, \mu')$ [9, Lemma 15].

**Remark 4.** *It is easy to verify that the two-colour analogues of Lemma 24* (i) *and* (iii) *hold.* $\diamondsuit$

We will need the following monotonicity facts about stable two-colour matchings. Similar results are proved in [8, Proposition 21], [5], and [15].

**Lemma 26.** *Let $\mu, \mu \in \mathbb{M}$ and assume that $\mu + \mu'$ is a simple point measure that is non-equidistant and has no descending chains. Let $m$ be the stable partial matching of $(\mu, \mu')$.*

*(i) Assume that $w \notin [\mu']$ and $\mu' + \delta_w$ is non-equidistant and has no descending chains. If $m'$ is the stable partial matching of $(\mu, \mu' + \delta_w)$, then*
$$\|z - m(z)\| \geq \|z - m'(z)\| \text{ for all } z \in [\mu].$$

*(ii) Let $x \in [\mu]$. If $m'$ is the stable partial matching of $(\mu - \delta_x, \mu')$, then*
$$\|z - m(z)\| \geq \|z - m'(z)\| \text{ for all } z \in [\mu - \delta_x]. \tag{23}$$

*Proof of Lemma 26.* Part (i) follows from [9, Lemma 17]. For part (ii), if $x$ is not matched under $m$, then $m' = m$, thus assume that $m(x) = y$. By Lemma 24 (i) and Remark 4, $m \backslash \{\{x, y\}\}$ is the unique stable partial matching for $(\mu - \delta_x, \mu' - \delta_y)$. Thus by part (i), $m'$, the unique stable matching for $(\mu - \delta_x, \mu')$, satisfies (23). $\square$

*Proof of Lemma 23.* The proof for the case when $\mathcal{R}$ is insertion-tolerant is given in [9, Theorem 6(i)]. In the case when $\mathcal{R}$ is deletion-tolerant

we proceed similarly to the proof of Lemma 22. Recall that in the two-colour case, $\mathcal{M}$ denotes the two-colour stable matching scheme for $\mathcal{R}$ and $\mathcal{B}$. Let $Z$ be a $\mathcal{B}$-point. Define $N(\mathcal{R}, Z)$ and $\mathcal{F}_1$ as in the proof of Lemma 22, so that $N(\mathcal{R}, Z)$ is the set of $\mathcal{R}$-points that would prefer $Z$ over their partners and $\mathcal{F}_1$ is the point process with support $N(\mathcal{R}, Z)$.

Towards a contradiction assume that $\mathbb{P}(\#N(\mathcal{R}, Z) < \infty) > 0$. There exists a unique stable partial matching for $(\mathcal{R} - \mathcal{F}_1, \mathcal{B})$ a.s.; denote it by $\mathcal{M}'$. From Lemma 26 (ii), it follows that

$$\mathbb{P}(N(\mathcal{R} - \mathcal{F}_1, Z) = 0) > 0. \tag{24}$$

From (24) and Remark 4 with Lemma 24 (iii), it follows that with positive probability, $\mathcal{M}' \setminus \{\{Z, \mathcal{M}'(Z)\}\}$ is the unique stable partial matching for $(\mathcal{R} - \mathcal{F}_1 - \mathcal{M}'(Z), \mathcal{B})$ and the $\mathcal{B}$-point $Z$ is left unmatched. By Lemma 15, there exists a Borel set $S$ with finite Lebesgue measure such that $\mathbb{P}(\mathcal{R}|_S = \mathcal{F}_1 + \delta_{\mathcal{M}'(Z)}) > 0$. By Theorem 1 (iii) and the deletion-tolerance of $\mathcal{R}$, we have that $\mathcal{R}|_{S^c} \prec \mathcal{R}$; furthermore, since $\mathcal{R}$ and $\mathcal{B}$ are independent, $(\mathcal{R}|_{S^c}, \mathcal{B}) \prec (\mathcal{R}, \mathcal{B})$. Thus with positive probability $(\mathcal{R}, \mathcal{B})$ has a stable partial matching with a unmatched $\mathcal{B}$-point. This contradicts the fact that $\mathcal{M}$ is the two-colour matching scheme for $\mathcal{R}$ and $\mathcal{B}$. □

## 7. Perturbed lattices and Gaussian zeros

7.1. **Low-fluctuation processes.** Propositions 11 and 12 will be proved using the following more general result, which states that processes satisfying various "low-fluctuation" conditions are neither insertion-tolerant nor deletion-tolerant. For a point process $\Pi$ and a measurable function $h : \mathbb{R}^d \to \mathbb{R}$ write

$$\Pi(h) := \int h(x) d\Pi(x) = \sum_{x \in [\Pi]} h(x). \tag{25}$$

Let $\overline{B}(0,1) := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ denote the closed unit ball.

**Proposition 27** (Low-fluctuation processes)**.** *Let $\Pi$ be a point process on $\mathbb{R}^d$ with finite intensity. Let $h : \mathbb{R}^d \to [0,1]$ be a measurable function with $h(x) = 1$ for all $x \in B(0, 1/2)$ and support in $\overline{B}(0,1)$. For each $n \in \mathbb{Z}^+$, set $h_n(x) := h(x/n)$ for all $x \in \mathbb{R}^d$.*

*(i) If $\Pi(h_n) - \mathbb{E}\Pi(h_n) \to 0$ in probability as $n \to \infty$, then $\Pi$ is neither insertion-tolerant nor deletion-tolerant.*

*(ii) If there exists a deterministic sequence $(n_k)$ with $n_k \to \infty$ such that*

$$\frac{1}{K} \sum_{k=1}^{K} \big(\Pi(h_{n_k}) - \mathbb{E}\Pi(h_{n_k})\big) \xrightarrow{\mathbb{P}} 0 \quad \text{as } K \to \infty, \tag{26}$$

*then $\Pi$ is neither insertion-tolerant nor deletion-tolerant.*

*(iii) Write* $N_n = \Pi(h_n) - \mathbb{E}\Pi(h_n)$*. If there exists a deterministic sequence* $(n_k)$ *with* $n_k \to \infty$ *and a discrete real-valued random variable* $N$ *such that for all* $\ell \in \mathbb{R}$*,*

$$\frac{1}{K}\sum_{k=1}^{K} \mathbf{1}[N_{n_k} \le \ell] \xrightarrow{\mathbb{P}} \mathbb{P}(N \le \ell) \quad \text{as } K \to \infty, \tag{27}$$

*then* $\Pi$ *is neither insertion-tolerant nor deletion-tolerant.*

In our application of Proposition 27 (iii), $N_n$ will be integer-valued (see (32) below).

*Proof of Proposition 27* (i). Let $m_n := \mathbb{E}\Pi(h_n)$. Since $\Pi(h_n) - m_n \to 0$ in probability, there exists a (deterministic) subsequence $n_k$ such that $\Pi(h_{n_k}) - m_{n_k} \to 0$ a.s. On the other hand, if $U$ is uniformly distributed in $B(0,1)$, then $(\Pi + \delta_U)(h_{n_k}) - m_{n_k} \to 1$ a.s. Therefore $\Pi$ is not insertion-tolerant. Similarly, if $Z$ any $\Pi$-point, then $(\Pi - \delta_Z)(h_{n_k}) - m_{n_k} \to -1$. So $\Pi$ is not deletion-tolerant. □

*Proof of Proposition 27* (ii). Suppose that (26) holds for some deterministic sequence $(n_k)$. Let $m_{n_k} := \mathbb{E}\Pi(h_{n_k})$, and for each integer $K > 0$ define $S_K : \mathbb{M} \to \mathbb{R}$ by

$$S_K(\mu) := \frac{1}{K}\sum_{k=1}^{K} (\mu(h_{n_k}) - m_{n_k}).$$

Thus $S_K(\Pi) \to 0$ in probability as $K \to \infty$, and there exists a subsequence $(K_i)$ so that $S_{K_i}(\Pi) \to 0$ a.s. However, if $U$ is uniformly distributed in $B(0,1)$, then $S_{K_i}(\Pi + \delta_U) \to 1$ a.s. Thus $\Pi$ cannot be insertion-tolerant. Similarly, if $Z$ is a $\Pi$-point, then $S_{K_i}(\Pi - \delta_Z) \to -1$ a.s. Thus $\Pi$ cannot be deletion-tolerant. □

*Proof of Proposition 27* (iii). Suppose that (27) holds for some deterministic sequence $(n_k)$ and some discrete random variable $N$. Let $m_{n_k} := \mathbb{E}\Pi(h_{n_k})$, and let $N_{n_k}(\mu) := \mu(h_n) - m_{n_k}$ for all $\mu \in \mathbb{M}$. For each integer $K > 0$, define $F_K : \mathbb{M} \times \mathbb{R} \to [0,1]$ by

$$F_K(\mu, \ell) := \frac{1}{K}\sum_{k=1}^{K} \mathbf{1}[N_{n_k}(\mu) \le \ell].$$

Thus $F_K(\Pi, \ell) \to \mathbb{P}(N \le \ell)$ in probability as $K \to \infty$ for all $\ell \in \mathbb{R}$. Since $N$ is discrete and has countable support, by a standard diagonal argument, there exists a subsequence $(K_i)$ so that $F_{K_i}(\Pi, \ell) \to \mathbb{P}(N \le \ell)$ a.s. for all $\ell \in \mathbb{R}$. Fix $a \in \mathbb{R}$ such that $\mathbb{P}(N \le a) \ne \mathbb{P}(N \le a+1)$. We have $F_{K_i}(\Pi, a) \to \mathbb{P}(N \le a)$ a.s. and $F_{K_i}(\Pi, a+1) \to \mathbb{P}(N \le a+1)$ a.s. However, if $U$ is uniformly distributed in $B(0,1)$, then $F_{K_i}(\Pi + \delta_U, a +$

$1) \to \mathbb{P}(N \le a)$ a.s. Thus $\Pi$ cannot be insertion-tolerant. Similarly, if $Z$ is a $\Pi$-point, then $F_{K_i}(\Pi - \delta_Z, a) \to \mathbb{P}(N \le a+1)$ a.s. Thus $\Pi$ cannot be deletion-tolerant. $\square$

7.2. **Gaussian zeros in the plane.**

*Proof of Proposition 12.* Let $\Upsilon_{\mathbb{C}}$ be the Gaussian zero process on the plane. Sodin and Tsirelson [21, Equation (0.6)] show that $\Upsilon_{\mathbb{C}}$ satisfies the conditions of Proposition 27 (i), with a twice differentiable function $h$; in particular they show that $\operatorname{Var} \Upsilon_{\mathbb{C}}(h_n) \to 0$ as $n \to \infty$. Hence $\Upsilon_{\mathbb{C}}$ is neither insertion-tolerant nor deletion-tolerant. $\square$

7.3. **Perturbed lattices in dimension** 2**.** The proof of Proposition 11 for the case $d = 2$ relies on the following lemma.

**Lemma 28.** *Let* $(Y_z : z \in \mathbb{Z}^2)$ *be i.i.d.* $\mathbb{R}^2$*-valued random variables with* $\mathbb{E}Y_0 = 0$ *and* $\operatorname{Var} \|Y_0\| = \sigma^2 < \infty$*. Let* $\Lambda$ *be the point process given by* $[\Lambda] := \{z + Y_z : z \in \mathbb{Z}^2\}$*. Let* $h : \mathbb{R}^2 \to [0,1]$ *have support in* $B(0,1)$*, and have Lipschitz constant at most* $c < \infty$*, and let* $h(x) = 1$ *for all* $x \in B(0, 1/2)$*. Define* $h_r(x) := h(x/r)$ *for* $x \in \mathbb{R}^2$ *and* $r > 0$*. Set* $m_r := \mathbb{E}\Lambda(h_r)$*.*

*(i) For all* $r > 0$ *we have* $\operatorname{Var} \Lambda(h_r) \le C$*, for some* $C = C(\sigma^2, c) < \infty$*.*
*(ii) For all* $r > 0$*, we have* $\operatorname{Cov}(\Lambda(h_r), \Lambda(h_R)) \to 0$ *as* $R \to \infty$*.*
*(iii) There exists a deterministic sequence* $(n_k)$ *with* $n_k \to \infty$ *such that* (26) *is satisfied with* $\Lambda$ *in place of* $\Pi$*; that is,*

$$\frac{1}{K} \sum_{k=1}^{K} \big(\Lambda(h_{n_k}) - \mathbb{E}\Lambda(h_{n_k})\big) \xrightarrow{\mathbb{P}} 0 \quad \textit{as } K \to \infty.$$

Lemma 28 parts (i) and (ii) will allow us to use a weak law of large numbers to prove (iii).

*Proof of Proposition 11 ($d = 2$).* We may clearly assume without loss of generality that $\mathbb{E}Y_0 = 0$. Now apply Lemma 28 (iii) together with Proposition 27 (ii). $\square$

*Proof of Lemma 28* (i)*.* Note that

$$\Lambda(h_r) = \sum_{z \in \mathbb{Z}^2} h_r(z + Y_z). \tag{28}$$

Thus by the independence of the $Y_z$, we have

$$\operatorname{Var} \Lambda(h_r) = \sum_{z \in \mathbb{Z}^2} \operatorname{Var} h_r(z + Y_z); \tag{29}$$

we will split this sum into two parts. We write $C_1, C_2$ for constants depending only on $\sigma^2$ and $c$.

Firstly, since $h_r$ has Lipschitz constant at most $c/r$, we have for all $z \in \mathbb{Z}^2$,

$$\operatorname{Var} h_r(z+Y_z) \le \mathbb{E}[(h_r(z+Y_z)-h_r(z))^2] \le \mathbb{E}[(c\|Y_z\|/r)^2] = (c\sigma/r)^2,$$

therefore

$$\sum_{z\in\mathbb{Z}^2:\|z\|\le 2r} \operatorname{Var} h_r(z+Y_z) \le C_1. \tag{30}$$

Secondly, since $h_r$ has support in $B(0,r)$,

$$\begin{aligned}\operatorname{Var} h_r(z+Y_z) &\le \mathbb{E}[h_r(z+Y_z)^2] \\ &\le \mathbb{P}[z+Y_z \in B(0,r)] = \mathbb{P}[Y_0 \in B(-z,r)],\end{aligned}$$

therefore

$$\begin{aligned}\sum_{z\in\mathbb{Z}^2:\|z\|>2r} \operatorname{Var} h_r(z+Y_z) &\le \sum_{z\in\mathbb{Z}^2:\|z\|>2r} \mathbb{P}[Y_0 \in B(-z,r)] \\ &\le C_2 r^2 \mathbb{P}(\|Y_0\|>r) \le C_2\sigma^2. \end{aligned}\tag{31}$$

The result now follows by combining (29)–(31). ☐

*Proof of Lemma 28* (ii). Note that by Lemma 28 (i), for all $r, R > 0$, we have that $\operatorname{Cov}(\Lambda(h_r),\Lambda(h_R)) < \infty$. By (28) and independence of the $Y_z$ we have

$$\begin{aligned}\operatorname{Cov}(\Lambda(h_r),\Lambda(h_R)) =& \mathbb{E}\Big(\sum_{z\in\mathbb{Z}^2} h_r(z+Y_z)\, h_R(z+Y_z)\Big) \\ &- \sum_{z\in\mathbb{Z}^2} \mathbb{E}h_r(z+Y_z)\, \mathbb{E}h_R(z+Y_z).\end{aligned}$$

Let $R > 2r$. If $h_r(z+Y_z) > 0$, then $h_R(z+Y_z) = 1$; thus

$$\operatorname{Cov}(\Lambda(h_r),\Lambda(h_R)) = m_r - \sum_{z\in\mathbb{Z}^2} \mathbb{E}h_r(z+Y_z)\, \mathbb{E}h_R(z+Y_z).$$

Since $h_R \uparrow 1$ as $R \to \infty$, for each $z \in \mathbb{Z}^2$ we have by the monotone convergence theorem that $\mathbb{E}h_R(z+Y_z) \uparrow 1$ as $R \to \infty$. An additional application of the monotone convergence theorem shows that

$$\lim_{R\to\infty} \sum_{z\in\mathbb{Z}^2} \mathbb{E}h_r(z+Y_z)\, \mathbb{E}h_R(z+Y_z) = \sum_{z\in\mathbb{Z}^2} \mathbb{E}h_r(z+Y_z) = m_r. \quad \square$$

We will employ the following weak law of large numbers for dependent sequences to prove Lemma 28 (iii).

**Lemma 29.** *Let $Z_1, Z_2, \ldots$ be real-valued random variables with finite second moments and zero means. If there exists a sequence $b(k)$ with $b(k) \to 0$ as $k \to \infty$ such that $\mathbb{E}(Z_n Z_m) \le b(n-m)$ for all $n \ge m$, then $(Z_1 + \cdots + Z_n)/n \to 0$ in probability as $n \to \infty$.*

Lemma 29 is a straightforward generalization of the standard $L^2$ weak law. See [4, Chapter 1, Theorem 5.2 and Exercise 5.2].

**Corollary 30.** *Let $Z_1, Z_2, \dots$ be real-valued random variables with finite second moments and zero means. Suppose that there exists $C > 0$, such that $\mathbb{E}|Z_m|^2 \le C$ for all $m \in \mathbb{Z}^+$. If for all $m \in \mathbb{Z}^+$ we have $\mathbb{E}(Z_m Z_n) \to 0$ as $n \to \infty$, then there exists an increasing sequence of positive of integers $(r_n)$ such that $(Z_{r_1} + \cdots + Z_{r_n})/n \to 0$ in probability as $n \to \infty$. Furthermore, for any further subsequence $(r_{n_k})$ we have $(Z_{r_{n_1}} + \cdots + Z_{r_{n_k}})/k \to 0$ in probability as $n \to \infty$.*

*Proof.* Consider the sequence $b(k) := 1/k$, where we set $b(0) = C$. We will show that there exists a sequence $r_k$ so that $\mathbb{E}(Z_{r_n} Z_{r_m}) \le 1/n$ for all $n > m$. Thus $Z_{r_k}$ satisfies the conditions of Lemma 29 with $b(k)$. We proceed by induction. Set $r_1 = 1$. Suppose that $r_2, \dots, r_{k-1}$ have already been defined and satisfy $\mathbb{E}(Z_{r_n} Z_{r_m}) \le 1/n$ for all $1 \le m < n \le k-1$. It follows from Lemma 28 (ii) that there exists an integer $R > 0$ such that $\mathbb{E}(Z_{r_m} Z_R) \le 1/k$ for all $1 \le m \le k-1$; set $r_k := R$. Furthermore, if $(r_{n_k})$ is a subsequence of $(r_n)$, we have that if $m < k$, then $\mathbb{E}(Z_{r_{n_m}} Z_{r_{n_k}}) \le 1/n_k \le 1/k$. Thus $Z_{r_{n_k}}$ satisfies the conditions of Lemma 29 with $b(k)$. ☐

*Proof of Lemma 28* (iii). For each $n \in \mathbb{Z}^+$, set $Z_n := \Lambda(h_n) - m_n$. By Lemma 28 parts (i) and (ii), $Z_n$ satisfies the conditions of Corollary 30. ☐

7.4. **Perturbed lattices in dimension 1.** The proof of Proposition 11 for the case $d = 1$ relies on the following lemma.

**Lemma 31.** *Let $(Y_z : z \in \mathbb{Z})$ be i.i.d. $\mathbb{R}$-valued random variables. Let $\Lambda$ be the point process given by $[\Lambda] := \{z + Y_z : z \in \mathbb{Z}\}$. Define $h(x) := \mathbf{1}_{(-1,1]}(x)$ for all $x \in \mathbb{R}$ and set $h_n(x) := h(x/n)$ for $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. For each $n \in \mathbb{Z}^+$, let $N_n := \Lambda(h_n) - \mathbb{E}\Lambda(h_n)$. Assume that $\mathbb{E}|Y_0| < \infty$.*

*(i) The family of random variables $(N_n)_{n \in \mathbb{Z}^+}$ is tight and integer-valued.*

*(ii) For any $k, \ell \in \mathbb{R}$ and $a \in \mathbb{Z}^+$,*

$$\mathbb{P}(N_a \le k, N_n \le \ell) - \mathbb{P}(N_a \le k)\, \mathbb{P}(N_n \le \ell) \to 0 \quad \text{as } n \to \infty.$$

*(iii) There exists a deterministic sequence $(n_k)$ with $n_k \to \infty$ and an integer-valued random variable $N$ such that (27) is satisfied; that is, for all $\ell \in \mathbb{R}$,*

$$\frac{1}{K} \sum_{k=1}^{K} \mathbf{1}[N_{n_k} \le \ell] \ \xrightarrow{\mathbb{P}}\ \mathbb{P}(N \le \ell) \quad \text{as } K \to \infty.$$

As in the case $d = 2$, Lemma 31 parts (i) and (ii) will allow us to use a weak law of large numbers to prove (iii). Let us note that the assumption that $\mathbb{E}|Y_0| < \infty$ is not necessary for Lemma 31 part (ii).

*Proof of Proposition 11 ($d = 1$).* Apply Lemma 31 (iii) together with Proposition 27 (iii). $\square$

*Proof of Lemma of 31* (i). The following simple calculation (an instance of the 'mass-transport principle') shows that $\mathbb{E}\Lambda(0,1] = 1$:

$$\mathbb{E}\Lambda(0,1] = \sum_{z\in\mathbb{Z}} \mathbb{P}\big(Y_z + z \in (0,1]\big) = \sum_{z\in\mathbb{Z}} \mathbb{P}\big(Y_0 \in (-z,-z+1]\big) = 1.$$

Thus

$$N_n = \Lambda(-n,n] - 2n \text{ for all } n \in \mathbb{Z}^+. \tag{32}$$

For $A, B \subseteq \mathbb{R}$, write

$$T_A^B := \#\{z \in A \cap \mathbb{Z} : z + Y_z \in B\};$$

that is, the number of $\Lambda$-points in $B$ that originated from $A$. Observe that for $n \in \mathbb{Z}^+$,

$$N_n = T_{(n,\infty)}^{(-n,n]} + T_{(-\infty,-n]}^{(-n,n]} - T_{(-n,n]}^{(n,\infty)} - T_{(-n,n]}^{(-\infty,-n]}. \tag{33}$$

On the other hand, $\mathbb{E}|Y_0| < \infty$ implies easily that $K_+ := \mathbb{E}T_{(-\infty,0]}^{[0,\infty)} < \infty$ and $K_- := \mathbb{E}T_{[0,\infty)}^{(-\infty,0]} < \infty$. By translation-invariance, each term on the right side of (33) is bounded in expectation by one of these constants; for instance: $\mathbb{E}T_{(n,\infty)}^{(-n,n]} \le \mathbb{E}T_{[n,\infty)}^{(-\infty,n]} = K_-$. Hence $\mathbb{E}|N_n| \le 2K_+ + 2K_-$ for all $n \in \mathbb{Z}^+$. $\square$

*Proof of Lemma 31* (ii). Let $\mathfrak{F}_n := \sigma(\{z + Y_z \in [-n,n]\} : z \in \mathbb{Z})$. We will show that for any event $E \in \sigma(Y_z : z \in \mathbb{Z})$, we have

$$\mathbb{P}(E \mid \mathfrak{F}_n) \to \mathbb{P}(E) \text{ a.s. as } n \to \infty. \tag{34}$$

From (34), the result follows, since $\{N_n \le \ell\} \in \mathfrak{F}_n$. It suffices to check (34) for $E$ in the generating algebra of events that depend on only finitely many of the $Y_z$. But for such an event, say $E \in \sigma(Y_z : -m \le z \le m)$, we observe that $\mathbb{P}(E \mid \mathfrak{F}_n)$ equals the conditional probability of $E$ given the *finite* $\sigma$-algebra $\sigma(\{z+Y_z \in [-n,n]\} : -m \le z \le m)$, hence the required convergence follows from an elementary computation. $\square$

*Proof of Lemma 31* (iii). By Lemma 31 (i) we may choose an integer-valued $N$ and a subsequence $(c_n)$ so that $N_{c_n} \xrightarrow{d} N$ as $n \to \infty$. We will

show that for all $\ell \in \mathbb{Z}$, there is a further subsequence $c_{n_k} =: r_k$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\Big[\mathbf{1}[N_{r_k} \le \ell] - \mathbb{P}(N_{r_k} \le \ell)\Big] \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty. \tag{35}$$

Clearly, the result follows from (35) and the fact that $N_{r_k} \xrightarrow{d} N$ as $k \to \infty$.

We use Corollary 30 in conjunction with a diagonal argument to prove (35). Consider an enumeration of the integers given by $\ell_1, \ell_2, \ldots$ For each $i \in \mathbb{Z}^+$, let $Z_k^i := \mathbf{1}[N_{c_k} \le \ell_i] - \mathbb{P}(N_{c_k} \le \ell_i)$. By Lemma 31 (ii) and Corollary 30, there exists a subsequence $c^1_{n_k} := r^1_k$ such that (35) holds with $r_k$ replaced by $r^1_k$, and $\ell$ replaced by $\ell_1$. Similarly, we may choose $(r^2_k)$ to be a subsequence of $(r^1_k)$ so that (35) holds with $r_k$ replaced by $r^2_k$, and $\ell$ replaced by $\ell_2$; moreover Corollary 30 assures us that (35) holds with $r_k$ replaced by $r^2_k$, and $\ell$ replaced by $\ell_1$. Similarly define the sequence $(r^i_k)$ for each $i \in \mathbb{Z}^+$. By taking the diagonal sequence $r_k := r^k_k$, we see that (35) holds for all $\ell \in \mathbb{Z}$. □

7.5. **Gaussian zeros in the hyperbolic plane.** The proof of Proposition 13 uses the following consequence of a result of Peres and Virág.

**Proposition 32.** *If $\Upsilon_{\mathbb{D}}$ is the Gaussian zero process on the hyperbolic plane and $\Upsilon^*_{\mathbb{D}}$ is its Palm version, then $\Upsilon^*_{\mathbb{D}} \prec \Upsilon_{\mathbb{D}}+\delta_0$ and $\Upsilon_{\mathbb{D}}+\delta_0 \prec \Upsilon^*_{\mathbb{D}}$.*

*Proof.* Let $\Upsilon_{\mathbb{D}}$ be the process of zeros of $\sum_{n=0}^{\infty} a_n z^n$, where the $a_n$'s are i.i.d. standard complex Gaussian random variables. Let $E_k$ be the event that $\Upsilon_{\mathbb{D}}(B(0,1/k)) > 0$. Peres and Virág [20, Lemma 18] prove that the conditional law of $(a_0, a_1, \ldots)$ given $E_k$ converges as $k \to \infty$ to the law of $(0, \widehat{a}_1, a_2, \ldots)$, where $\widehat{a}_1$ is independent of the $a_n$'s, and has a rotationally symmetric law with $|\widehat{a}_1|$ having probability density $2r^3e^{-r^2}$.

Let $\widehat{\Upsilon}_{\mathbb{D}}$ be the process of zeros of the power series with coefficients $(0, \widehat{a}_1, a_2, \ldots)$. Since the latter sequence is mutually absolutely continuous in law with $(0, a_1, a_2 \ldots)$, we have that $\widehat{\Upsilon}_{\mathbb{D}}$ and $\Upsilon_{\mathbb{D}}+\delta_0$ are mutually absolutely continuous in law.

By Rouché's theorem from complex analysis [6, Ch. 8, p. 229], the above convergence implies that the conditional law of $\Upsilon_{\mathbb{D}}$ given $E_k$ converges to the law of $\widehat{\Upsilon}_{\mathbb{D}}$ (the convergence is in distribution with respect to the vague topology for point processes). By [12, Theorem 12.8] it follows that $\widehat{\Upsilon}_{\mathbb{D}} \stackrel{d}{=} \Upsilon^*_{\mathbb{D}}$. □

*Proof of Proposition 13.* It follows from Proposition 32 and Theorems 4 and 5 with Remark 1 that the Gaussian zero process on the hyperbolic plane is insertion-tolerant and deletion-tolerant. □

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## References


[1] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.*, 121(3):501–505, 1989.

[2] D. J. Daley and G. Last. Descending chains, the lilypond model, and mutual-nearest-neighbour matching. *Adv. in Appl. Probab.*, 37(3):604–628, 2005.

[3] D. J. Daley and D. Vere-Jones. *An Introduction to the Theory of Point Processes. Vol. II.* Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[4] R. Durrett. *Probability: Theory and Examples.* Duxbury Press, Belmont, CA, second edition, 1996.

[5] D. Gale and L. Shapley. College admissions and stability of marriage. *Amer. Math. Monthly*, 69:9–15, 1962.

[6] T. W. Gamelin. *Complex analysis.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.

[7] O. Häggström and R. Meester. Nearest neighbor and hard sphere models in continuum percolation. *Random Structures Algorithms*, 9(3):295–315, 1996.

[8] C. Hoffman, A. E. Holroyd, and Y. Peres. A stable marriage of Poisson and Lebesgue. *Ann. Probab.*, 34(4):1241–1272, 2006.

[9] A. E. Holroyd, R. Pemantle, Y. Peres, and O. Schramm. Poisson matching. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45(1):266–287, 2009.

[10] A. E. Holroyd and Y. Peres. Extra heads and invariant allocations. *Ann. Probab.*, 33(1):31–52, 2005.

[11] J. B. Hough, M. Krishnapur, Y. Peres, and B. Virág. *Zeros of Gaussian analytic functions and determinantal point processes*, volume 51 of *University Lecture Series.* American Mathematical Society, Providence, RI, 2009.

[12] O. Kallenberg. *Random Measures.* Akademie-Verlag, Berlin, third edition, 1983.

[13] O. Kallenberg. *Foundations of Modern Probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[14] O. Kallenberg. Invariant measures and disintegrations with applications to Palm and related kernels. *Probab. Theory Related Fields*, 139(1-2):285–310, 2007.

[15] D. E. Knuth. *Stable Marriage and its Relation to Other Combinatorial Problems*, volume 10 of *CRM Proceedings & Lecture Notes.* American Mathematical Society, Providence, RI, 1997. An introduction to the mathematical analysis of algorithms, Translated from the French by Martin Goldstein and revised by the author.

[16] G. Last. Modern random measures: Palm theory and related models. In *New perspectives in stochastic geometry.* Clarendon Press, Oxford, 2008.

[17] G. Last. Stationary random measures on homogeneous spaces. *Journal of Theoretical Probability*, 2009.

[18] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[19] R. Meester and R. Roy. *Continuum Percolation*, volume 119 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[20] Y. Peres and B. Virág. Zeros of the i.i.d. Gaussian power series: a conformally invariant determinantal process. *Acta Math.*, 194(1):1–35, 2005.

[21] M. Sodin and B. Tsirelson. Random complex zeroes. I. Asymptotic normality. *Israel J. Math.*, 144:125–149, 2004.

[22] H. Thorisson. Transforming random elements and shifting random fields. *Ann. Probab.*, 24:2057–2064, 1996.

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